

## Boundary Integral Formulation of Domain Variables for Three-Dimensional Initial Strain Problems

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### Abstract

The boundary-type integral formulations of domain variables are presented explicitly for three-dimensional initial strain problems. The domain variables were first represented by complete series of polynomial expansions, then the related domain integrals are transformed into the boundary ones with the aid of the intrinsic correlation among the integral kernels as well as the higher order fundamental solutions. The applicability of the formulation is discussed briefly.

### INTRODUCTION

The boundary element method has pronounced merits characterized by dividing elements only at the boundaries and with high precision of solutions, which reduces data preparation and saves computing time<sup>1</sup>. The internal cells in the domain, however, are usually indispensable if the problem under consideration is inhomogeneous or nonlinear. There have been many attempts and developments in order to overcome this limitations in recent years, such as the dual reciprocity method<sup>2</sup>, the multiple reciprocity method<sup>3</sup> and the computing point method<sup>4-5</sup>. One of the key points in the multiple reciprocity method is the transformation of domain integrals into boundary-type via higher order fundamental solutions<sup>2</sup>. In the computing point method, the inhomogeneous term in the nonlinear Poisson equation was represented by polynomial expansions and then the domain integrals were transformed into the boundary-type therefore the problem can be solved without domain discretizations<sup>4-5</sup>. All of these works show the effectiveness of the transformation in getting rid of the domain discretization.

In a general sense, the initial strain problems include problems such as the thermo-elasticity, the residual stress problem<sup>6</sup>, the creep problem and the elasto-plasticity in initial strain algorithm. Some of them belong to the material nonlinearity. The applicability of the procedure lies in how well the domain variables, the initial strains, can be described by polynomial approximation in terms of the space coordinates and how the transformation can be realized from domain-type into boundary one.

In the previous work, the domain variables were assumed to be represented by polynomials and the transformation from domain into boundary-type integrals was realized via the intrinsic correlation among the integral kernels for two-dimensional problem<sup>7</sup>. The present work extended the transformation to three-dimensional case, which

will be realized, similarly, with the aid of both the intrinsic correlation among the integral kernels and the higher order fundamental solutions. The formulations are given explicitly and discussed briefly.

## BASIC EQUATIONS

The initial strain problems can be described by two boundary integral equations, the displacement equation and the stress equation for a domain  $\Omega$  with boundary  $\Gamma$  as follows<sup>1</sup>:

$$\frac{1}{2}u_i(p) = \int_{\Gamma} \tau_j(q)u_{ij}^*(p, q)d\Gamma(q) - \int_{\Gamma} u_j(q)\tau_{ij}^*(p, q)d\Gamma(q) + \int_{\Omega} \varepsilon_{jk}(q)\sigma_{ijk}^*(p, q)d\Omega(q) \quad (1)$$

$$\sigma_{ij}(p) = \int_{\Gamma} \tau_k(q)u_{ijk}^*(p, q)d\Gamma(q) - \int_{\Gamma} u_k(q)\tau_{ijk}^*(p, q)d\Gamma(q) + \int_{\Omega-\Omega_\varepsilon} \varepsilon_{kl}(q)\sigma_{ijkl}^*(p, q)d\Omega(q) + \varepsilon_{kl}(p)O_{ijkl}(p) \quad (2)$$

where

$p$  —source point

$q$  —field point

$\Omega_\varepsilon$  —tiny zone centered at  $p$

$u_j$  —displacement

$\tau_j$  —traction at the boundary

$\sigma_{ij}$  —stress

$\varepsilon_{ij}$  —initial strain or initial strain increment

$u_{ij}^*$  —fundamental solution of elasticity

$\tau_{ij}^*, \sigma_{ijk}^*, u_{ijk}^*, \tau_{ijk}^*, \sigma_{ijkl}^*, O_{ijkl}$  —derived fundamental solutions

In the above two equations there are domain integrals of the initial strains generated owing to material nonlinearity. If the initial strain  $\varepsilon_{ij}$  is replaced by the term  $\alpha\theta\delta_{ij}$  the two equations can be used for thermo-elasticity, where  $\alpha$  is the thermal expansion coefficient of the material and  $\theta$  is the temperature. The effects of the domain variables on the displacement and the stress equations have conventionally to be evaluated by numerical calculations over divided internal cells in the domain. Suppose the initial strain can be described by polynomial approximation in terms of the spatial coordinates as follows:

$$\varepsilon_{ij}(q) \approx \sum_{m=0, n=0, l=0}^{m+n+l=N} C_{ij}^{mnl} x_1^m(q)x_2^n(q)x_3^l(q) \quad (3)$$

where  $N$  is the highest order and  $C_{ij}^{mnl}$  the coefficients to be determined of the polynomials. Insert the above expression into the displacement and the stress equations,

respectively, and replace the spatial coordinates by the two-point variables between the source and the field points:

$$x_k = x_k(q) - x_k(p) \quad (4)$$

then any term of the polynomial expansions for the domain integrals in the displacement and the stress equations can be expressed as

$$C_{jk}^{mnl} \int_{\Omega} x_1^m(q) x_2^n(q) x_3^l(q) \sigma_{ijk}^*(p, q) d\Omega(q) = C_{jk}^{mnl} \sum_{s=0}^m \sum_{t=0}^n \sum_{u=0}^l \frac{m!n!l!}{(m-s)!s!(n-t)!t!(l-u)!u!} \times \\ \times [x_1(p)]^{m-s} [x_2(p)]^{n-t} [x_3(p)]^{l-u} \int_{\Omega} x_1^s x_2^t x_3^u \sigma_{ijk}^*(p, q) d\Omega(q) \quad (5)$$

$$C_{jk}^{mnl} \left\{ \int_{\Omega} x_1^m(q) x_2^n(q) x_3^l(q) \sigma_{ijkl}^*(p, q) d\Omega(q) + x_1^m(p) x_2^n(p) x_3^l(p) O_{ijkl}(p) \right\} = \\ = C_{jk}^{mnl} \left\{ \sum_{s=0}^m \sum_{t=0}^n \sum_{u=0}^l \frac{m!n!l!}{(m-s)!s!(n-t)!t!(l-u)!u!} [x_1(p)]^{m-s} [x_2(p)]^{n-t} [x_3(p)]^{l-u} \times \right. \\ \left. \times \int_{\Omega} x_1^s x_2^t x_3^u \sigma_{ijk}^*(p, q) d\Omega(q) + x_1^m(p) x_2^n(p) x_3^l(p) O_{ijkl}(p) \right\} \quad (6)$$

respectively, where  $m, n, l, s, t$  and  $u$  are all integers. The transformation in the above two equations is linear and performed with a short subroutine. The domain integrals in the right hand side of equations (5) and (6) can then be transformed into the boundary integrals via the correlation between the kernels<sup>7</sup>, the Cauchy's integral relation and the equilibrium equation shown as follows:

$$\sigma_{ijk}^*(p, q) n_k(q) = \tau_{ij}^*(p, q) \quad (7)$$

$$\sigma_{ijk,k}^*(p, q) + \delta_{ij} \delta(p, q) = 0 \quad (8)$$

where  $n_k(q)$  is the outward normal. The derived fundamental solution  $\sigma_{ijk}^*(p, q)$  is the stress at  $q$  when there is a unit force acting at point  $p$  in  $i$  direction. And  $\tau_{ij}^*(p, q)$  represents the corresponding traction at the same point.

## FOFMULATIONS FOR DISPLACEMENT EQUATION

The derived fundamental solutions  $\sigma_{ijk}^*(p, q)$  in the domain integral of the displacement equation is presented here explicitly in the three-dimensional form for clarity:

$$\sigma_{ijk}^*(p, q) = \frac{1}{8\pi(1-\nu)r} \left\{ (1-2\nu)(\delta_{jk} r_{,i} - \delta_{ki} r_{,j} - \delta_{ij} r_{,k}) - 3r_{,i} r_{,j} r_{,k} \right\} \quad (9)$$

where  $\nu$  is Poisson's ratio and  $r$  the distance between the source and the field points

$$r = \sqrt{x_k x_k} \quad (10)$$

The boundary-type formulation of the zeroth order term of the polynomials is as follows:

$$\int_{\Omega} \sigma_{ijk}^*(p, q) d\Omega(q) = \int_{\Gamma} x_k \tau_{ij}^*(p, q) d\Gamma(q) \quad (11)$$

which has been derived under the state with a definite physical meaning, that is, the state of a traction free domain with a uniformly distributed unit initial strain over the domain<sup>7</sup>. This equation can be extended to a general form with the Cauchy's relation (7) and the equilibrium equation (8) as follows:

$$\int_{\Omega} [m x_1^{m-1} x_2^n x_3^l \delta_{1k} + n x_1^m x_2^{n-1} x_3^l \delta_{2k} + l x_1^m x_2^n x_3^{l-1} \delta_{3k}] \sigma_{ijk}^*(p, q) d\Omega(q) = \int_{\Gamma} x_1^m x_2^n x_3^l \tau_{ij}^*(p, q) d\Gamma(q) \quad (12)$$

Then all the boundary-type formulations, except a special one, of integral kernels of the three-dimensional initial strain problem can be obtained explicitly:

$$\int_{\Omega} x_1^m x_2^n x_3^l \sigma_{111}^* d\Omega(q) = \frac{1}{m+n+l+1} \int_{\Gamma} x_1^{m+1} x_2^n x_3^l \tau_{11}^* d\Gamma(q) \quad (13)$$

$$\int_{\Omega} x_1^m x_2^n x_3^l \sigma_{222}^* d\Omega(q) = \frac{1}{m+n+l+1} \int_{\Gamma} x_1^m x_2^{n+1} x_3^l \tau_{22}^* d\Gamma(q) \quad (14)$$

$$\int_{\Omega} x_1^m x_2^n x_3^l \sigma_{333}^* d\Omega(q) = \frac{1}{m+n+l+1} \int_{\Gamma} x_1^m x_2^n x_3^{l+1} \tau_{33}^* d\Gamma(q) \quad (15)$$

$$\int_{\Omega} x_1^m x_2^n x_3^l \sigma_{112}^* d\Omega(q) = \frac{1}{m+n+l+1} \int_{\Gamma} x_1^m x_2^{n+1} x_3^l \tau_{11}^* d\Gamma(q) \quad (16)$$

$$\int_{\Omega} x_1^m x_2^n x_3^l \sigma_{223}^* d\Omega(q) = \frac{1}{m+n+l+1} \int_{\Gamma} x_1^m x_2^n x_3^{l+1} \tau_{22}^* d\Gamma(q) \quad (17)$$

$$\int_{\Omega} x_1^m x_2^n x_3^l \sigma_{331}^* d\Omega(q) = \frac{1}{m+n+l+1} \int_{\Gamma} x_1^{m+1} x_2^n x_3^l \tau_{33}^* d\Gamma(q) \quad (18)$$

$$\int_{\Omega} x_1^m x_2^n x_3^l \sigma_{113}^* d\Omega(q) = \frac{1}{m+n+l+1} \int_{\Gamma} x_1^m x_2^n x_3^{l+1} \tau_{11}^* d\Gamma(q) \quad (19)$$

$$\int_{\Omega} x_1^m x_2^n x_3^l \sigma_{221}^* d\Omega(q) = \frac{1}{m+n+l+1} \int_{\Gamma} x_1^{m+1} x_2^n x_3^l \tau_{22}^* d\Gamma(q) \quad (20)$$

$$\int_{\Omega} x_1^m x_2^n x_3^l \sigma_{332}^* d\Omega(q) = \frac{1}{m+n+l+1} \int_{\Gamma} x_1^m x_2^{n+1} x_3^l \tau_{33}^* d\Gamma(q) \quad (21)$$

$$\int_{\Omega} x_1^m x_2^n x_3^l \sigma_{122}^* d\Omega(q) = \frac{1}{n+1} \int_{\Gamma} x_1^m x_2^{n+1} x_3^l \tau_{12}^* d\Gamma(q) - \frac{m}{(n+1)(m+n+l+1)} \int_{\Gamma} x_1^{m-1} x_2^{n+2} x_3^l \tau_{11}^* d\Gamma(q) - \frac{l}{n+1} \int_{\Omega} x_1^m x_2^{n+1} x_3^{l-1} \sigma_{123}^* d\Omega(q) \quad (22)$$

$$\int_{\Omega} x_1^m x_2^n x_3^l \sigma_{233}^* d\Omega(q) = \frac{1}{l+1} \int_{\Gamma} x_1^m x_2^n x_3^{l+1} \tau_{23}^* d\Gamma(q) - \frac{n}{(l+1)(m+n+l+1)} \int_{\Gamma} x_1^m x_2^{n-1} x_3^{l+2} \tau_{22}^* d\Gamma(q) - \frac{m}{l+1} \int_{\Omega} x_1^{m-1} x_2^n x_3^{l+1} \sigma_{123}^* d\Omega(q) \quad (23)$$

$$\int_{\Omega} x_1^m x_2^n x_3^l \sigma_{311}^* d\Omega(q) = \frac{1}{m+1} \int_{\Gamma} x_1^{m+1} x_2^n x_3^l \tau_{31}^* d\Gamma(q) - \frac{l}{(m+1)(m+n+l+1)} \int_{\Gamma} x_1^{m+2} x_2^n x_3^{l-1} \tau_{33}^* d\Gamma(q) - \frac{n}{m+1} \int_{\Omega} x_1^{m+1} x_2^{n-1} x_3^l \sigma_{123}^* d\Omega(q) \quad (24)$$

$$\int_{\Omega} x_1^m x_2^n x_3^l \sigma_{133}^* d\Omega(q) = \frac{1}{l+1} \int_{\Gamma} x_1^m x_2^n x_3^{l+1} \tau_{13}^* d\Gamma(q) - \frac{m}{(l+1)(m+n+l+1)} \int_{\Gamma} x_1^{m-1} x_2^n x_3^{l+2} \tau_{11}^* d\Gamma(q) - \frac{n}{l+1} \int_{\Omega} x_1^m x_2^{n-1} x_3^{l+1} \sigma_{123}^* d\Omega(q) \quad (25)$$

$$\int_{\Omega} x_1^m x_2^n x_3^l \sigma_{211}^* d\Omega(q) = \frac{1}{m+1} \int_{\Gamma} x_1^{m+1} x_2^n x_3^l \tau_{21}^* d\Gamma(q) - \frac{n}{(m+1)(m+n+l+1)} \int_{\Gamma} x_1^{m+2} x_2^{n-1} x_3^l \tau_{22}^* d\Gamma(q) - \frac{l}{m+1} \int_{\Omega} x_1^{m+1} x_2^n x_3^{l-1} \sigma_{123}^* d\Omega(q) \quad (26)$$

$$\int_{\Omega} x_1^m x_2^n x_3^l \sigma_{322}^* d\Omega(q) = \frac{1}{n+1} \int_{\Gamma} x_1^m x_2^{n+1} x_3^l \tau_{32}^* d\Gamma(q) - \frac{l}{(n+1)(m+n+l+1)} \int_{\Gamma} x_1^m x_2^{n+2} x_3^{l-1} \tau_{33}^* d\Gamma(q) - \frac{m}{n+1} \int_{\Omega} x_1^{m-1} x_2^{n+1} x_3^l \sigma_{123}^* d\Omega(q) \quad (27)$$

All of the above formulations have a similar form to those in the two-dimensional case<sup>7</sup>, except for a special term  $\sigma_{123}^*$  of the domain integral which appears at the right hand side in some of these formulations (Eqns.(22)-(27)). They can be transformed via the higher order fundamental solutions of elasticity as shown below.

### FORMULATION OF THE SPECIAL TERM

Let us start with the Galerkin tensor of the zeroth order for the three-dimensional elasticity<sup>1</sup>:

$$G_{ij}^* = G_{ij}^{*(0)} = \frac{r}{8\pi\mu} \delta_{ij} \quad (28)$$

where  $\mu$  is the shear modulus. It is not difficult to write out the  $N$ -th ordered Galerkin tensor for elasticity:

$$G_{ij}^{*(N)} = \frac{r^{2N+1}}{4\pi\mu(2N+2)!} \delta_{ij} \quad (N = 0, 1, 2, \dots) \quad (29)$$

which has the recurrence relation of

$$G_{ij,kk}^{*(N)} = G_{ij}^{*(N-1)} \quad (30)$$

Then the corresponding higher order fundamental solution or Kelvin's solution can be deduced:

$$\begin{aligned} u_{ij}^{*(N)} &= G_{ij,kk}^{*(N)} - \frac{1}{2(1-\nu)} G_{ik,kj}^{*(N)} \\ &= \frac{r^{2N-1}}{16\pi\mu(1-\nu)(N+1)(2N)!} \left\{ [4(1-\nu)(N+1)-1] \delta_{ij} - (2N-1)r_i r_j \right\} \\ &\quad (N = 0, 1, 2, \dots) \end{aligned} \quad (31)$$

With the constitutive relation of elasticity, the corresponding higher order derived fundamental solution in the displacement equation can therefore be obtained:

$$\begin{aligned} \sigma_{ijk}^{*(N)} &= \mu \left[ \beta \delta_{jk} u_{il,l}^{*(N)} + (u_{ij,k}^{*(N)} + u_{ik,j}^{*(N)}) \right] \\ &= \frac{r^{2N-2}}{16\pi(1-\nu)N(N+1)(2N-2)!} \left\{ [2(1-\nu)(N+1)-1] (\delta_{ij} r_k + \delta_{ki} r_j) \right. \\ &\quad \left. + [2\nu(N+1)-1] \delta_{jk} r_i - (2N-3)r_i r_j r_k \right\} \quad (N = 1, 2, \dots) \end{aligned} \quad (32)$$

where  $\beta$  is a material constant defined by

$$\beta = 2\nu / (1 - 2\nu) \quad (33)$$

It is noticed that the equilibrium equation does no longer hold for the derived fundamental solution in their higher order counterparts, but it has the following simple form:

$$\sigma_{ijk,k}^{*(N)} = \frac{r^{2N-3}}{4\pi(2N-2)!} \delta_{ij} \quad (N = 1, 2, \dots) \quad (34)$$

And the outward normal directional derivative of the derived fundamental solution is

$$\begin{aligned} \frac{\partial}{\partial n} \sigma_{ijk}^{*(N)} &= \frac{r^{2N-3}}{16\pi(1-\nu)N(N+1)(2N-2)!} \left\{ (2N-3) \frac{\partial r}{\partial n} \left( [2(1-\nu)(N+1)-1] (\delta_{ij} r_k + \delta_{ki} r_j) \right) \right. \\ &\quad \left. + [2\nu(N+1)-1] \delta_{jk} r_i - (2N-5)r_i r_j r_k \right\} - (2N-3)(r_j r_k n_i + r_k r_i n_j + r_i r_j n_k) \\ &\quad + [2(1-\nu)(N+1)-1] (\delta_{ij} n_k + \delta_{ki} n_j) + [2\nu(N+1)-1] \delta_{jk} n_i \quad (N = 1, 2, \dots) \end{aligned} \quad (35)$$

The transformation of the special term from the domain-type into the boundary one can then be realized by the recurrence formula using Gaussian divergence theorem:

$$\begin{aligned} \int_{\Omega} x_1^m x_2^n x_3^l \sigma_{ijk}^{*(N)} d\Omega(q) &= \int_{\Omega} x_1^m x_2^n x_3^l \sigma_{ijk,ss}^{*(N+1)} d\Omega(q) \\ &= \int_{\Gamma} x_1^m x_2^n x_3^l \frac{\partial}{\partial n} \sigma_{ijk}^{*(N+1)} d\Gamma(q) - \int_{\Gamma} \frac{\partial}{\partial n} (x_1^m x_2^n x_3^l) \sigma_{ijk}^{*(N+1)} d\Gamma(q) + \int_{\Omega} (x_1^m x_2^n x_3^l)_{,ss} \sigma_{ijk}^{*(N+1)} d\Omega(q) \end{aligned} \quad (N = 0, 1, 2, \dots) \quad (36)$$

The explicit forms of the two high ordered kernels of the special term are:

$$\sigma_{123}^{*(N)} = \frac{-(2N-3)r^{2N-2}}{16\pi(1-\nu)N(N+1)(2N-2)!} r_{,1} r_{,2} r_{,3} \quad (37)$$

$$\frac{\partial}{\partial n} \sigma_{123}^{*(N)} = \frac{-(2N-3)r^{2N-3}}{16\pi(1-\nu)N(N+1)(2N-2)!} \left[ (2N-5) \frac{\partial r}{\partial n} r_{,1} r_{,2} r_{,3} + r_{,1} r_{,2} n_3 + r_{,2} r_{,3} n_1 + r_{,3} r_{,1} n_2 \right] \quad (N = 1, 2, \dots) \quad (38)$$

With these formulations, all of the domain integrals have been represented by the boundary-type for the displacement equation.

Formulation of the stress equations, not shown here, are also available as the same scheme mentioned above.

## DISCUSSIONS

The applicability of the procedure in getting rid of domain discretizations by polynomial expansions lies in how the transformation can be realized from domain-type into boundary one and how well the domain variables, the initial strains, can be described by polynomial approximation in terms of the field coordinates. In the present work, the field variable  $x_k(q)$  has been replaced by the two point variable  $x_k$  in the integrals through transformations (5) and (6), therefore the intrinsic correlation between the kernels can be made use of to avoid the term by term integrals except the special term in the three-dimensional case.

As stated above, the transformations (5) and (6) can be performed via a short subroutine. There are other merits, for example, the term  $O_{ijkl}$  owing to the singularity appears only in the zeroth order or constant term of the polynomials and will be embraced in the boundary integral without special consideration so that the programming can be simplified. Even for the zeroth order or constant term of the polynomials, the order of singularity in the boundary-type integrals is reduced by one. And because these formulations hold no matter where the source point locates, inside or outside the domain or on the boundary, they might be used to evaluate the domain integrals with strong singularities when domain discretization is performed.

In performing the computing point methods<sup>4,5</sup>, the polynomial can be expanded to any order conveniently using these formulations to meet the precision requirements. The

formulations can be reduced to deal with the problem of thermo-elasticity by replacing the initial strain  $\varepsilon_{ij}$  with the term  $\alpha\theta\delta_{ij}$ , for example, suppose

$$\alpha\theta H^*_i(p, q) = \alpha\theta\delta_{jk}\sigma^*_{ijk}(p, q) \quad (39)$$

$$\alpha\theta H^*_{ij}(p, q) = \alpha\theta\delta_{kl}\sigma^*_{ijkl}(p, q) \quad (40)$$

where  $H^*_i$  and  $H^*_{ij}$  stand for the domain integral kernels in the displacement and the stress equations, which can be derived by the combination of  $\delta_{jk}\sigma^*_{ijk}$  and  $\delta_{kl}\sigma^*_{ijkl}$ , respectively, for the problem of thermo-elasticity. Then the boundary-type formulations for the zeroth order term can be obtained as follows:

$$\int_{\Omega} H^*_i(p, q)d\Omega(q) = \int_{\Gamma} x_j\tau^*_{ij}(p, q)d\Gamma(q) \quad (62)$$

$$\int_{\Omega} H^*_{ij}(p, q)d\Omega(q) + \delta_{kl}O_{ijkl}(p) = \int_{\Gamma} x_k\tau^*_{ijk}(p, q)d\Gamma(q) \quad (63)$$

For residual stress problems<sup>6</sup>, the initial strains formed by material nonlinearity can be approximated by polynomials and expected to be solved without domain discretizations.

## CONCLUSION

With the aid of the intrinsic correlation among the integral kernels as well as the higher order fundamental solutions, the explicit form of boundary-type integral was deduced for the three-dimensional initial strain problems aiming at boundary element analysis in which the domain variables were represented by complete series of polynomial expansions.

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