

HIGHLY ACCURATE CRACK TIP ANALYSIS

Anh-Vũ Phan, L.J. Gray and T. Kaplan

Computer Science and Mathematics Division
Oak Ridge National Laboratory, TN, USA.

Glaucio H. Paulino

Department of Civil and Environmental Engineering
University of Illinois at Urbana-Champaign, IL, USA.

Abstract

The ability to do crack propagation simulations relies on computing accurate stress intensity factors (SIFs) at crack tips. For many years, the standard quarter-point shape functions have been widely used for crack tip elements in finite and boundary element analyses. However, experience has shown that while the quarter-point produces reasonably accurate K_I (opening) SIF, it is far less successful in predicting K_{II} (sliding) and K_{III} (twisting). As a result, it is more difficult to accurately predict curved crack growth. In this paper, we present an adjustment to the quarter-point crack tip element and employ this element in symmetric-Galerkin boundary integral fracture analysis. The standard quarter-point element is modified so that the near-tip crack opening displacement (COD) satisfies a known constraint, namely that the term linear in the distance to the tip must vanish. In test cases where the SIFs are known exactly, this modified quarter-point element, in conjunction with the Displacement Correlation Technique (DCT), has produced significantly more accurate results than the standard element.

1. Introduction

The numerical modeling of crack stability and propagation requires the accurate determination of SIFs, which in turn relies on the correct representation of the local crack tip stress and displacement fields. For a crack geometry, William's (1952, 1957) result for the COD Δu_k , $k = 1, 2$ in the neighborhood of the tip is:

$$\Delta u_k(r, \theta) = b_k(\theta)r^{\frac{1}{2}} + c_k(\theta)r + d_k(\theta)r^{\frac{3}{2}} + \dots \quad (1)$$

where r , θ are the distance to, and the direction emanating from, the tip, respectively.

In both finite and boundary element modeling of discrete cracks, the standard approach consists of incorporating the critical \sqrt{r} behavior by means of the 'quarter-point' element originally developed by Henshell and Shaw (1975), and Barsoum (1976). Use of special elements at the crack tip has significantly improved the accuracy of SIF calculations (*e.g.*, Blandford *et al.* 1981), and the original quarter-point element has been extended and refined in a number of ways (*e.g.*, Banks-Sills 1991). Nevertheless, in either finite or boundary element analyses, the prediction of K_{II} and K_{III} has not been nearly as accurate as for K_I .

Recently, Gray and Paulino (1998) proved that, irrespective of the problem geometry or boundary conditions, the series expansion in Eq. (1) must have $c_k = 0$ for $\Delta \mathbf{u}$ on the crack surface (for related work see Martin, 1991). As will be discussed below, in general the quarter-point element fails to satisfy this constraint, and it is this flaw that is addressed in this paper. By forcing the linear term in $\Delta \mathbf{u}$ to be zero, we expect to have a more accurate analysis in the tip region. This is in fact borne out by the test calculations, as a significant improvement in the accuracy of stress intensity factors is seen. These calculations employ a symmetric-Galerkin boundary integral fracture analysis (see, for example, Sirtori *et al.* 1992 and Bonnet *et al.* 1998) coupled with the DCT for evaluating the SIFs.

2. Modified Quarter-Point Element and SIFs

The two-dimensional quarter-point element is based upon the three-noded quadratic element. For $t \in [0, 1]$, the shape functions for this element are given by

$$\begin{aligned}\psi_1(t) &= (1-t)(1-2t), \\ \psi_2(t) &= 4t(1-t), \\ \psi_3(t) &= t(2t-1).\end{aligned}\tag{2}$$

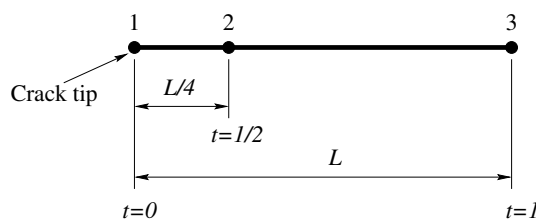


Figure 1: Crack tip element.

As $\Delta \mathbf{u} = 0$ at the crack tip, which is assumed to be at $t = 0$ (Fig. 1), the representations of the crack tip geometry and COD are

$$\Gamma(t) = \sum_{j=1}^3 (x_j \psi_j(t), y_j \psi_j(t)), \quad \Delta u_k(t) = \sum_{j=2}^3 \left(\Delta u_1^j \psi_j(t), \Delta u_2^j \psi_j(t) \right)\tag{3}$$

where (x_j, y_j) are the coordinates of the three nodes defining the element, and Δu_k^j the nodal values of the COD.

By moving the mid-node coordinates (x_2, y_2) three-fourths of the way towards the tip, the parameter t becomes $\sqrt{r/L}$, with L being the distance from (x_1, y_1) to (x_3, y_3) (Henshell and Shaw, 1975; Barsoum, 1976). As a consequence, the leading order term in Δu_k^j at $t = 0$, which is t , is the correct square root of distance. Note however, that the next term, which is t^2 , is r/L . According to Gray and Paulino (1998), this term should vanish, and the modification presented below replaces this term with $(r/L)^{3/2}$.

For the new approximation, we keep the representation of $\Gamma(t)$ as in Eq. (3), so that the property $t \approx \sqrt{r}$ remains. However for the COD, we define new shape functions by adding a cubic term:

$$\begin{aligned}\hat{\psi}_2(t) &= 4t(1-t) - \frac{4}{3}t(1-t)(1-2t) = -\frac{8}{3}(t^3 - t), \\ \hat{\psi}_3(t) &= t(2t-1) + \frac{2}{3}t(1-t)(1-2t) = \frac{1}{3}(4t^3 - t).\end{aligned}\quad (4)$$

This additional contribution accomplishes the cancellation of the $t^2 \approx r$ term, while maintaining the Kronecker delta property of the interpolation $\hat{\psi}_j(t_i) = \delta_{ji}$.

The general expression of SIFs by means of the DCT is given by

$$K_I = \frac{G}{\kappa + 1} \lim_{r \rightarrow 0} \sqrt{\frac{2\pi}{r}} \Delta u_2, \quad K_{II} = \frac{G}{\kappa + 1} \lim_{r \rightarrow 0} \sqrt{\frac{2\pi}{r}} \Delta u_1 \quad (5)$$

where Δu_k is the COD in the coordinate system associated with the crack tip under consideration, G is shear modulus, ν is Poisson's ratio, and $\kappa = 3 - 4\nu$ (plane strain) or $\kappa = (3 - \nu)/(1 + \nu)$ (plane stress).

Use of the modified shape functions (Eq. (4)) in Eq. (3), and then substitution of the resulting Δu_k in (5), with $t = \sqrt{r/L}$, yields

$$K_I = \frac{G}{3(\kappa + 1)} \sqrt{\frac{2\pi}{L}} (8\Delta u_2^2 - \Delta u_2^3), \quad K_{II} = \frac{G}{3(\kappa + 1)} \sqrt{\frac{2\pi}{L}} (8\Delta u_1^2 - \Delta u_1^3). \quad (6)$$

Thus, SIFs are given directly in terms of the nodal values of the COD at the crack tip element.

3. Symmetric Galerkin Boundary Integral Fracture Analysis

The test calculations reported below employ a 2-D symmetric-Galerkin boundary integral approximation, and thus this section presents a very brief overview of this technique. A good introduction to the Galerkin method is provided in the recent text by Bonnet (1995), and a recent review by Bonnet *et al.* (1998) on symmetric-Galerkin provides an excellent summary and references to the literature.

The 2-D displacement boundary integral equation is given by (Rizzo 1967)

$$U(P) = u_k(P) + \int_{\partial B} T_{kj}(P, Q) u_j(Q) dQ - \int_{\partial B} U_{kj}(P, Q) \tau_j(Q) dQ = 0, \quad (7)$$

where \mathbf{u} and $\boldsymbol{\tau}$ denote displacement and traction, respectively. This equation is for an interior point P , and hence the coefficient of the "free term" is one. The singular integrals are interpreted as a limit from the interior. As customary, the kernel functions $T_{kj}(P, Q)$ and $U_{kj}(P, Q)$ are given by the Kelvin solution for a point load in an infinite medium. The corresponding integral equation for the surface stress is essential for treating crack geometries, and in the symmetric-Galerkin approach it is this equation that is employed on the crack surface. The

stress equation is obtained by differentiating Eq. (7) with respect to P , resulting in

$$\mathcal{S}(P) = \sigma_{lk}(P) + \int_{\partial B} S_{lkm}(P, Q)u_m(Q) dQ - \int_{\partial B} D_{lkm}(P, Q)\tau_m(Q) dQ = 0 . \quad (8)$$

In a Galerkin formulation, the above integral equations are enforced ‘on average’, in the form

$$\int_{\partial B} \psi_l(P)\mathcal{U}(P) dP = 0 , \quad \int_{\partial B} \psi_l(P)\mathcal{S}(P) dP = 0 . \quad (9)$$

The weight functions $\psi_l(P)$ are the shape functions employed to interpolate the boundary displacements and tractions. Note that the modified shape functions are employed to define the Galerkin equations in the crack tip region.

In regards to the singular integration, it is important to note that the modified shape functions is easily incorporated in an existing quadratic element code. The standard quadratic and modified quarter-point shape functions can be written as

$$\hat{\psi}_j(t) = \psi_j(t) + \alpha \beta_j t(1-t)(1-2t) . \quad (10)$$

Here α is used to distinguish a crack tip element ($\alpha = 1$) from a regular element ($\alpha = 0$); the values for the constants β are $\beta_1 = \beta_3 = \pm 2/3$ and $\beta_2 = \mp 4/3$, and the upper or lower sign is employed if the crack tip is located at the first ($t = 0$) or third node ($t = 1$), respectively. In the latter case, by letting $s = 1 - t$, one again obtains the modified shape functions in Eq. (4) in terms of variable s .

From this form it is clear that implementing the modified crack tip only requires the additional integration of the $t(1-t)(1-2t)$ expression. As this function is zero at the three nodes, it only contributes to the “lower order” singular terms, and in these terms it simply changes some polynomial coefficients.

4. Test Calculations

Two crack examples are considered in this Section. For all examples, consistent units are used. SIFs, provided by the modified and standard quarter-point (QP) elements, and calculated by means of the DCT, are compared. The point here is to assess the quality of the modified quarter point element by means a very simple method such as the DCT.

4.1 Pair of interacting circular-arc cracks

It is generally recognized that the DCT combined with the standard QP shape functions usually produce poor accuracy for SIFs in case of mixed-mode situations. Use of the modified QP element instead can improve considerably this accuracy as shown in this example.

Consider a pair of circular-arc cracks of radius $R = 0.1$, angle $\theta = \pi/2$, and subjected to remote biaxial tension σ in an unbounded domain as shown in Fig. 2. Five elements (of uniform length) per crack are used.

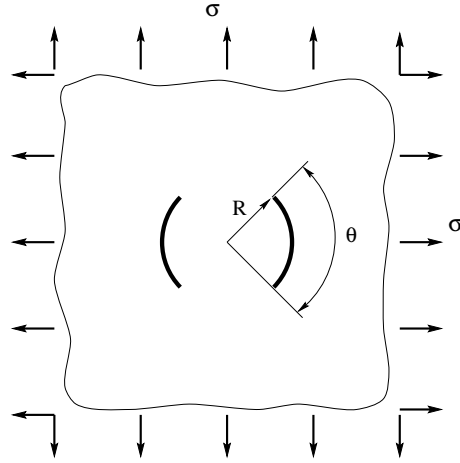


Figure 2: Pair of circular-arc cracks.

Numerical results normalized by the exact solution given by Tada *et al.* (1973), are shown in Table 1. Both K_I and K_{II} provided by the modified QP element are much accurate then those from the standard one.

Normalized SIFs	Standard QP	Modified QP
K_I/K_I^{exact}	1.0711	1.0047
$K_{II}/K_{II}^{\text{exact}}$	0.7512	0.9625

Table 1: Normalized SIFs.

The effect of crack interaction on K_I and K_{II} (as a function of θ) is depicted in Fig. 3. For K_I , the numerical error increases as the gap between the cracks decreases. However, it can be seen that the increase in error rate for the standard QP is higher as the cracks come closer to each other. For K_{II} , while the result obtained from the standard QP is mostly unchanged with the increase of θ , the modified QP element helps to dramatically improve this quantity, even where the crack tips are very close ($\theta \approx \pi$).

4.2 Three-point beam bending

This example deals with a surface crack and bending deformation (Fig. 4). K_I is evaluated for a wide range of geometry of the beam and crack using crack elements of uniform length $\Delta a = 0.02W$. The SG-BEM results are compared with the analytical solution proposed by Guinea *et al.* (1998) and listed in Table 2. Deviation of the standard and modified QP results from the reference is illustrated in Fig. 5. It can be seen that all results obtained from the modified QP shape functions are in better agreement with the reference except in case $L/W = 2.5$ and $a/W = 0.1$. Since the Guinea *et al.*'s solution is a good but not an absolutely exact

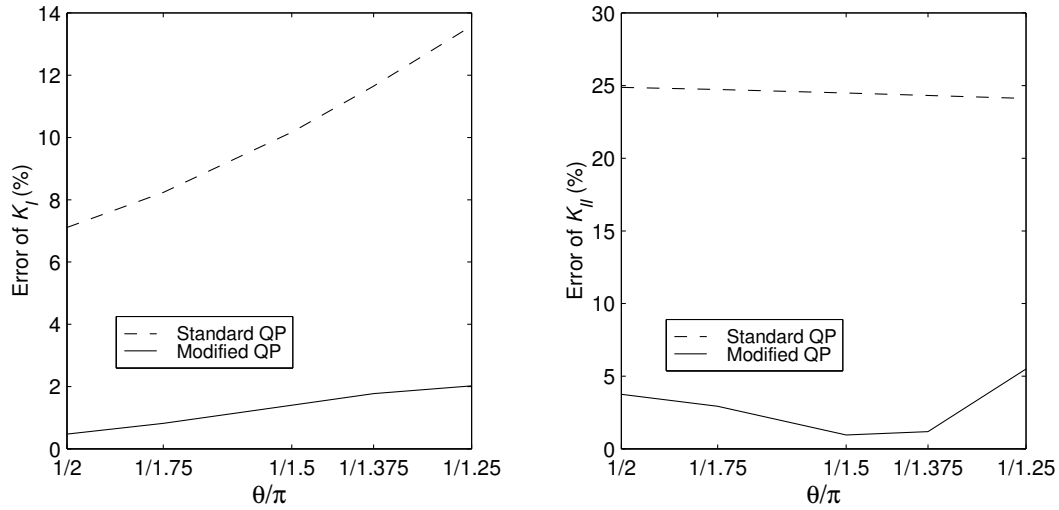


Figure 3: Effect of crack interaction on SIFs.

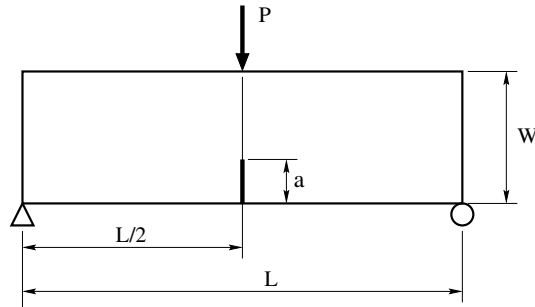


Figure 4: Three-point bend beam.

reference, this incompatibility might serve as an indication to revise the proposed expression in the above case.

5. Conclusions

The modified QP crack tip element defined in this paper has been shown to produce highly accurate SIFs. This is the case even though the calculations employed relatively coarse meshes, combined with a simple *local* method (DCT) to calculate SIFs. Moreover, the modified QP element has been shown to be consistently superior to the standard QP element. Thus, this improved element should be very useful for crack propagation simulations, as the meshing in these calculations cannot be minutely controlled. This investigation, as well as the extension of this work to three dimensions, is currently being pursued by the authors.

a/W	L/W = 2.5		L/W = 8	
	Standard QP	Modified QP	Standard QP	Modified QP
0.1	0.9898	0.9870	1.0020	0.9981
0.2	0.9816	0.9855	0.9959	0.9995
0.3	0.9800	0.9868	0.9941	1.0011
0.4	0.9797	0.9887	0.9914	1.0011
0.5	0.9791	0.9905	0.9868	0.9990
0.6	0.9770	0.9916	0.9791	0.9946
0.7	0.9722	0.9920	0.9657	0.9862
0.8	0.9610	0.9908	0.9546	0.9851

Table 2: K_I/K_I^{ref} as functions of L/W and a/W

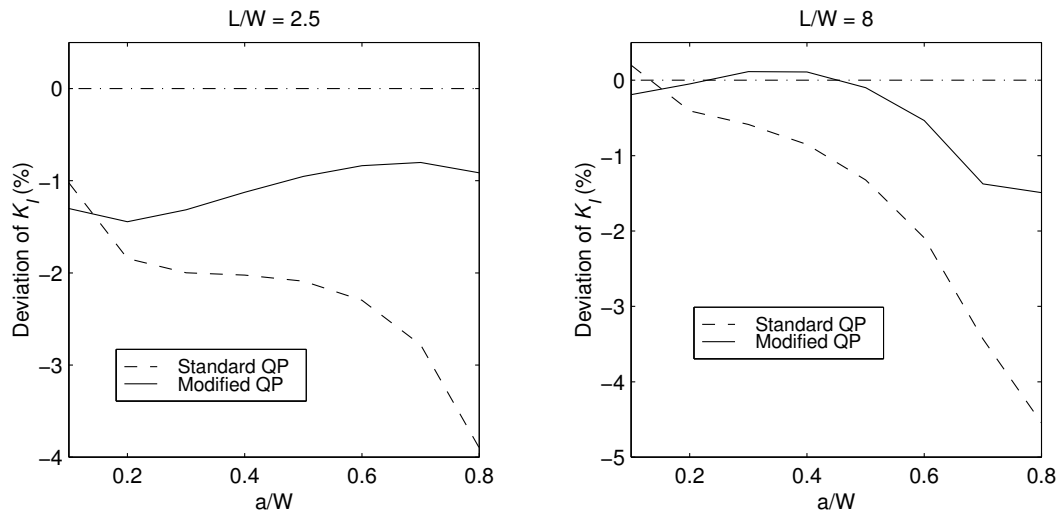


Figure 5: Deviation of the standard and modified QP results from the reference.

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