Boundary Integral Equation Formulations for Piezoelectric Solids Containing Thin Shapes

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Abstract

A weakly-singular form of the piezoelectric boundary integral equation (BIE) formulation is introduced in this paper, which eliminates the calculation of any singular integrals in the piezoelectric BIE. The crucial question of whether or not the piezoelectric BIE will degenerate when applied to crack and thin shell-like problems is discussed. It can be shown analytically that the conventional BIE for piezoelectricity does degenerate for crack problems, but does *not* degenerate for thin piezoelectric shells. The latter has significant implications in applications of the piezoelectric BIE to analyze thin piezoelectric films used widely as sensors and actuators. Implementation of the boundary element method (BEM) for the developed piezoelectric BIE in analyzing thin piezoelectric solids, including treatment of the nearly-singular integrals existing in such cases, are discussed and two numerical examples are presented.

Introduction

Piezoelectric materials can be used as sensors and actuators in smart materials or micro-electro-mechanical systems (MEMS), because they have many desirable properties (see, e.g., Ref. [1]). An electric current is induced in a piezoelectric solid when it is applied with a mechanical load. This property can be used to design sensors to monitor the deformation of a structure. Conversely, a mechanical deformation is produced when it is applied with a voltage. This property can be used to design actuators to control vibrations or noises of a structure. Simulations of piezoelectric solids, on the other hand, are very difficult, because of the anisotropy in piezoelectric materials, coupling of elastic and electric fields, and thinness of the piezoelectric devices (the thickness of sensors/actuators is in the range of a few μm to a few hundred μm). Accurate 3-D modeling and analysis should be employed for such analysis.

In 3-D analysis, the BIE/BEM approach has been demonstrated to be a viable alternative to the finite element method (FEM) for many problems, due to its features of surface-only discretization and high accuracy in stress and fracture analyses. Another advantage of the BIE/BEM, which was recognized only in recent years, is its high accuracy and efficiency in handling thin shell-like structures, layered structures (multi-coatings or thin films), thin voids or open cracks [2-8]. It has been demonstrated that the BIE/BEM can handle the various thin-body problems very effectively, regardless of the thinness of the structures or voids, or non-uniform thickness, as long as the nearly-singular integrals are computed accurately [3, 4, 9, 10]. Much fewer boundary elements can be used to solve these problems for which the number of required finite elements is at least two-orders larger to achieve the same accuracy in stress analysis [3-6]. Considering the fact that the piezoelectric sensors and actuators are often made in thin shapes (films or patches), the BIE/BEM with thin body capabilities has the potential to provide a very efficient and accurate tool in the analysis of such piezoelectric materials.

In this paper, the recent development in the BIE formulation for piezoelectric materials and the BEM implementation is summarized. The weakly-singular form of the piezoelectric BIE is introduced, which can eliminate the calculation of any singular integrals in the BEM. The crucial question of whether or not the piezoelectric BIE will degenerate when applied to crack and thin shell-like structures is discussed. It can be shown that the conventional BIE for piezoelectricity does degenerate for crack problems, but does *not* degenerate for shell-like structures, in the limit as the two opposing surfaces approaching each other. The latter has significant implications in the applications of the piezoelectric BIE to piezoelectric films used widely as sensors and actuators in smart materials and MEMS. How to deal with the nearly-singular integrals in the piezoelectric BIE when they are applied to thin shapes (thin voids, open cracks, or thin films and coatings) are discussed. Numerical tests to show the degeneracy of the piezoelectric BIE for crack problems, and the non-degeneracy and accuracy in analyzing thin piezoelectric solids using the developed piezoelectric BEM, are also presented.

The Boundary Integral Equations for Piezoelectricity

Consider a piezoelectric solid occupying domain V with boundary S. The basic equations governing the elastic and electric fields in a linear piezoelectric material can be written as follows (see, e.g., Refs. [11]) (index notation is used here).

Equilibrium Equations:

$$\sigma_{ij,j} + f_i = 0, \tag{1}$$

$$D_{i,i} - q = 0, \tag{2}$$

where σ_{ij} is the stress tensor, f_i the body force vector per unit volume, D_i the electric displacement vector and q the intrinsic electric charge per unit volume.

Constitutive Equations:

$$\sigma_{ij} = C_{ijkl}s_{kl} - e_{kij}E_k, \qquad \text{(converse effect)} \tag{3}$$

$$D_i = e_{ikl}s_{kl} + \varepsilon_{ik}E_k, \qquad \text{(direct effect)} \tag{4}$$

where s_{kl} is the strain tensor, E_k the electric field, C_{ijkl} the elastic modulus tensor measured in a constant electric field, e_{ijk} the piezoelectric tensor and ε_{ij} the dielectric tensor measured at constant strains.

Strain and Electric Fields:

$$s_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{5}$$

$$E_i = -\phi_{,i}, \tag{6}$$

where u_i is the elastic displacement vector and ϕ the electric potential.

Boundary Conditions (BCs):

$$t_i = \sigma_{ij} n_j = \bar{t}_i, \quad \text{on } S_t, \qquad u_i = \bar{u}_i, \quad \text{on } S_u; \quad \text{(mechanical BCs)} \tag{7}$$

$$\omega = -D_i n_i = \bar{\omega}, \quad \text{on } S_\omega, \qquad \phi = \bar{\phi}, \quad \text{on } S_\phi; \quad \text{(electric BCs)} \tag{8}$$

where t_i is the traction, ω the surface charge, n_i the unit outward normal vector and the barred quantities indicate given values. Note that $S = S_t \cup S_u = S_\omega \cup S_\phi$.

Applying the integral identity for the piezoelectric fundamental solution developed in Ref. [12], we can obtain the following *weakly-singular form of* the boundary integral equation for 2-D piezoelectricity problems [12]:

$$\begin{aligned} \int_S \mathbf{T}(P,P_0)[\mathbf{u}(P) - \mathbf{u}(P_0)]dS(P) &= \int_S \mathbf{U}(P,P_0)\mathbf{t}(P)dS(P) \\ &+ \int_V \mathbf{U}(P,P_0)\mathbf{b}(P)dV(P), \qquad \forall P_0 \in S, \end{aligned} \tag{9}$$

for a *finite* piezoelectric solid (cf., the potential and elasticity cases [13, 14]), in which,

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ -\phi \end{Bmatrix}, \qquad \mathbf{t} = \begin{Bmatrix} t_1 \\ t_2 \\ -\omega \end{Bmatrix}, \qquad \mathbf{b} = \begin{Bmatrix} f_1 \\ f_2 \\ -q \end{Bmatrix},$$

$$\mathbf{U} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix}, \qquad \mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}, \tag{10}$$

are the *extended* displacement, traction and body force vectors, and the *extended* displacement and traction kernels in the 2-D piezoelectric fundamental solutions, respectively; P_0 the source point and P the field point. Note that the dimensions of matrices are 3×3 for 2-D piezoelectric problems because of the coupling of the elastic and electric fields.

The weakly-singular BIE (9) for piezoelectric solids has several advantages, compared with the singular BIE. There are no singular integrals in the weakly-singular BIE and its discretization leads directly to the conclusion that the diagonal terms can be determined by summing the off-diagonal terms for the matrix involving the singular kernel **T** (see, Ref. [13]). By employing weakly-singular BIE (9), one does not need to evaluate any jump terms explicitly in deriving the piezoelectric BIEs.

It is shown in Ref. [12] that the piezoelectric BIE (9) does degenerate when applied to the two opposing surfaces (S^+ and S^-) of a *crack*, but does not degenerate when applied to the two surfaces (again S^+ and S^-) of a *thin shell-like structure*. For example, using the matrix notation, Eq. (9) yields the following two equations in the limit as the opening $h \to 0$, for a crack-like problem [12]:

$$D^+(\mathbf{u}^+ - \mathbf{u}^-) + \frac{1}{2}(\mathbf{u}^+ + \mathbf{u}^-) = B^+(\mathbf{t}^+ + \mathbf{t}^-), \qquad \text{from } P_o \in S^+;$$

$$D^+(\mathbf{u}^+ - \mathbf{u}^-) + \frac{1}{2}(\mathbf{u}^+ + \mathbf{u}^-) = B^+(\mathbf{t}^+ + \mathbf{t}^-), \qquad \text{from } P_o \in S^-;$$

where $\mathbf{u}^+, \mathbf{t}^+$, $\mathbf{u}^-$ and $\mathbf{t}^-$ are the extended displacement and traction vectors, on S^+ and S^- , respectively, D^+ and B^+ are matrices or integral operators associated with the **U** and **T** kernels, respectively. The above two equations are exactly the same equation. Therefore, the piezoelectric BIE *does degenerate* when applied to cracks. On the other hand, for a thin piezoelectric shell, Eq. (9) leads to the following two equations in the limit as the thickness of the shell $h \to 0$ [12]:

$$D^+(\mathbf{u}^+ - \mathbf{u}^-) + \frac{1}{2}(\mathbf{u}^+ - \mathbf{u}^-) = B^+(\mathbf{t}^+ + \mathbf{t}^-), \qquad \text{from } P_o \in S^+;$$

$$D^+(\mathbf{u}^+ - \mathbf{u}^-) - \frac{1}{2}(\mathbf{u}^+ - \mathbf{u}^-) = B^+(\mathbf{t}^+ + \mathbf{t}^-), \qquad \text{from } P_o \in S^-;$$

which are two distinctive equations no matter how thin the shell is, *as long as* the piezoelectric shell is under *realistic* boundary conditions (e.g., no rigid-body motions, or not constrained on the entire boundary *S*; see discussions in [3]). Therefore, the piezoelectric BIE *does not degenerate* in general when applied to thin shells.

However, before one can apply BIE (9) to thin piezoelectric shell-like materials or structures, one has to deal with the nearly-singular integrals [3, 4, 9, 10] existing in such applications. As discussed in Ref. [15], the treatment of the nearly-singular integral for the 2-D piezoelectric BIE case is reduced to the following integral:

$$\int_{\Delta L} T_{ij}(P, P_0) dL(P) = \sum \int_{-1}^{1} \frac{a\xi^3 + b\xi^2 + c\xi + d}{e\xi^4 + f\xi^3 + g\xi^2 + h\xi + q} d\xi \ , \tag{11}$$

for a quadratic element ΔL, which is a summation of 24 integrals with parameters *a* to *q* being lengthy expressions containing the nodal coordinates and piezoelectric material constants. Either numerical integrations or analytical integrations (using, e.g., a symbolic software) can be employed to compute these integrals.

Numerical Examples

To show the *degeneracy* of the piezoelectric BIE for *crack-like* problems, a square piezoelectric medium (PZT-4, under plane strain condition) with an elliptical hole at the center is considered (Fig. 1). The domain is sufficiently large compared with the hole (*L*/*b* = 20) so that the analytical solution in Ref. [16] for an infinite domain can be used to validate the BEM results. The model is under uniform tension in the x_1 direction and the elliptical holes are formed by scaling a circular hole in the x_1 direction (axis a).

Very good BEM results can be obtained for stresses, mechanical displacements and electric displacements, if the ratio a/b is above 0.01, as reported in [12]. However, for the case $a/b = 0.001$, even the finest BEM mesh (204 elements) cannot provide converged results. Fig. 2 shows the electric displacement results in this case with increasing numbers of elements. The BEM results (using single-domain approach) do not converge and the symmetry in the BEM results with respect to the crack tip ($\theta = 90°$) is also lost (Fig. 2). This is a clear indication of the degeneracy of the piezoelectric BIE/BEM for crack-like problems, as predicted by the theory. A quick solution to this degeneracy is to use the multi-domain BEM approach for crack problems (see, e.g., Ref. [17]). Very good results are obtained in this piezoelectric BEM case as shown by the solid circles in Fig. 2.

To show the *non-degeneracy* of the piezoelectric BIE for *thin shell-like* problems, the same piezoelectric (PZT-5) strip example as presented in Ref. [18] is studied (Fig. 3). The polarization direction of the piezoelectric strip is in the x_3 direction. The exact solution for this problem is listed as Eqs. (87-89) in Ref. [18] (where the symbol "h" in Eq. (88) should be replaced by "$L/2$") and is employed here to validate the developed piezoelectric BEM for thin shell-like cases. A total of 16 boundary elements (four on each edge) are used in the BEM analysis. The thickness $2h$ is changed from L to $10^{-4}L$, with $L = 1.0$ mm. All the other parameters and boundary conditions are the same as given in Ref. [18]. The shear deformation dominates in this case, due to the applied voltages along the two vertical edges [18].

Figure 4 shows the results of the displacement component w along the bottom edge using the developed piezoelectric BEM for thin shell-like problems, as compared with the analytical solutions, for three different values of the thickness $2h$. The BEM results are very accurate and stable for the thin strip cases, without the need to increase the number of the boundary elements at all. This is a distinctive advantage of the developed piezoelectric BEM with thin-body capabilities over other BEM and FEM approaches which, in general, require a large number of elements in order to maintain the accuracy and stability in the analysis of these thin-body problems. More test problems on thin piezoelectric films and coatings using the developed BEM can be found in Ref. [15].

Conclusions

The important issues regarding the degeneracy of piezoelectric BIE for crack-like problems, and the non-degeneracy for thin shell-like piezoelectric structures, are discussed in this paper. Numerical examples to support the theoretical assertions are presented. The BEM results for the thin piezoelectric structures clearly demonstrate the high accuracy and efficiency of the developed BEM, which can provide a robust analysis tool for the design of smart materials and MEMS.

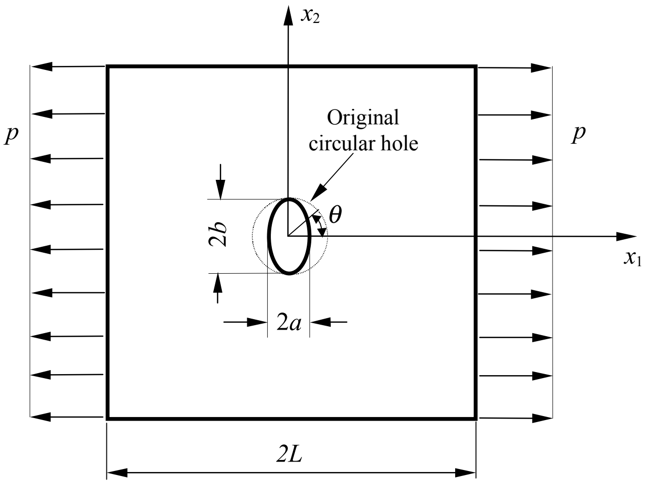

Figure 1. An elliptical hole in a square piezoelectric medium under tension.

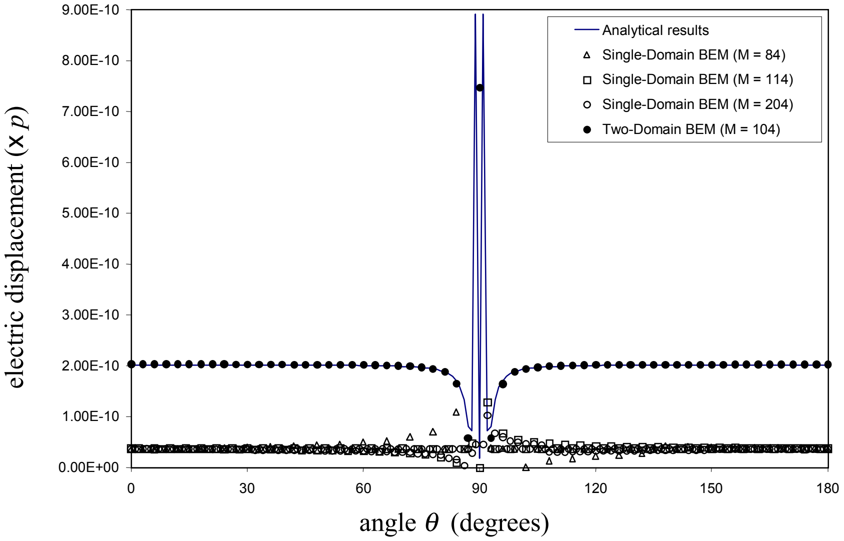

Figure 2. Degeneracy of the piezoelectric BIE for crack-like problems: Results for the magnitude of electric displacement on the edge of the elliptical hole when $a/b = 0.001$ (M = number of elements).

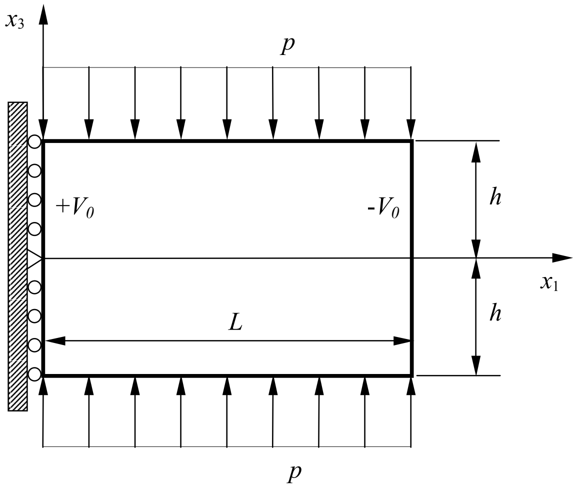

Figure 3. A PZT-5 strip subjected to a pressure load p and voltage V_0.

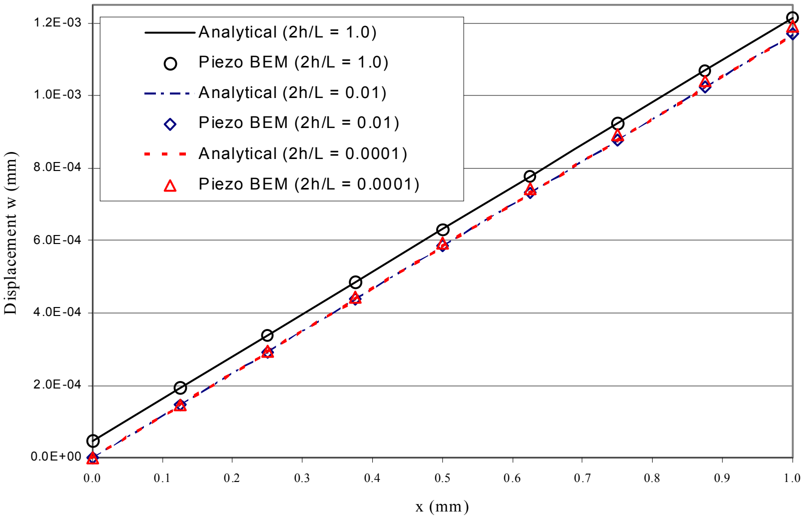

Figure 4. Non-degeneracy of the piezoelectric BIE for thin shell-like problems: Displacement component w along the bottom edge of the strip using the piezoelectric BEM (with 24 boundary elements).

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